Spring Algorithms and Symmetry*

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Abstract

Spring algorithms are regarded as effective tools for visualizing undirected graphs. One major feature of applying spring algorithms is to display symmetric properties of graphs. This feature has been confirmed by numerous experiments. In this paper, firstly we formalize the concepts of graph symmetries in terms of "reflectional" and "rotational" automorphisms; and characterize the types of symmetries, which can be displayed simultaneously by a graph layout, in terms of "geometric" automorphism groups. We show that our formalization is complete. Secondly, we provide general theoretical evidence of why many spring algorithms can display graph symmetry. Finally, the strength of our general theorem is demonstrated from its application to several existing spring algorithms.

1 Introduction

Graphs are commonly used in Computer Science to model relational structures such as programs, databases, and data structures. A good graph "layout" gives a clear understanding of a structural model; a bad graph layout is simply misleading. Automatic graph layout has received a great deal of attention since last decade. Many aesthetics for modelling goodness of graph layouts have been proposed from various applications, as well as graph layout algorithms for achieving those proposed layout aesthetics [4, 13]. In this paper, we are interested only in graph layouts in two dimensional Euclidean space.

Symmetry is one of the most important aesthetics for delivering an easily understandable graph layout. A graph with a symmetric structure requires us only to concentrate on its "core" part, while the other parts can be symmetrically mirrored from the symmetric structure. For instance, the graph layout in Figure 1(a) is much easier to be understood than that in Figure 1(b), though the graphs in Figures 1(a) and 1(b) are the same. A display of graph symmetry, thus, become crucially important for producing good graph layouts.

Several fast polynomial time algorithms have been reported for symmetrically drawing some special classes of graphs, such as *planar* graphs [10], *outerplanar* graphs [17], trees [16], *rooted*

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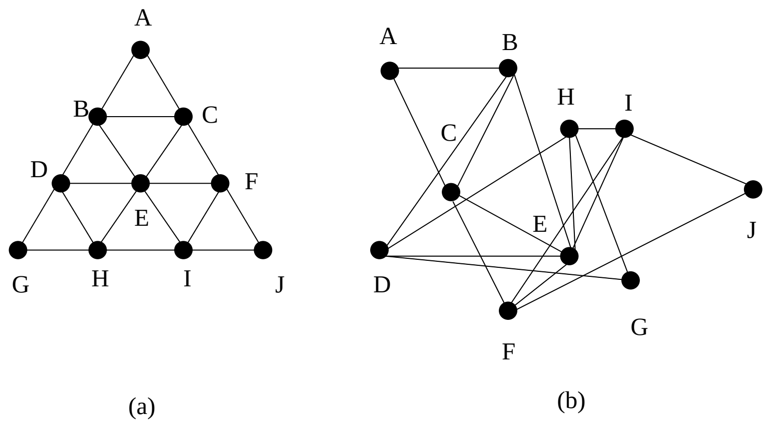

Figure 1: Two different layouts

trees [19], *s-t upward planar* graphs [5], *series-parallel* directed graphs [11], *well connected* hierarchically planar graphs [7], and etc. However, a display of graph symmetries has been shown NP-complete in general, even restricted only to a single symmetry [15]; the simultaneous display of a set of symmetries seems even harder. On the other hand, it has been demonstrated by numerous experiments that "spring (or force-directed) algorithms" [4] can serve as very effective heuristics for displaying graph symmetries; for instance, the layout in Figure 1(a) is produced by the spring algorithm in [12].

A spring algorithm takes a graph as input, simulates a system of springs defined on the graph, and outputs a locally minimum energy configuration. A number of spring algorithms ([2, 3, 6, 8, 12, 23]) have been developed. For example, the algorithm of [12] places a spring f_{uv} between each pair of vertices u, v; the *natural length* of f_{uv} is assigned as the *graph-theoretic length* (the length of the shortest path) between u and v.

Symmetries of a graph correspond to "automorphisms" (see Section 2) of the graph; but not every automorphism can be displayed as a symmetry of a graph layout. For instance, in Figure 2(a), the automorphism $p = (1 \to 2 \to 3 \to 1)(4 \to 5 \to 4)$ cannot be displayed as a symmetry of any graph layout. Further, a pair of displayable automorphisms are not always simultaneously displayable. For instance, Figure 2(b) displays the "rotational" automorphism $p_1 = (1 \to 2 \to 3 \to 4 \to 1)$, but does not display the rotational automorphism $p_2 = (1 \to 2 \to 3 \to 1)$ with vertex 4 fixed. On the other hand, Figure 2(c) displays p_2, but does not display p_1. In fact, as we show in Section 2, there is no graph layout which simultaneously displays p_1 and p_2.

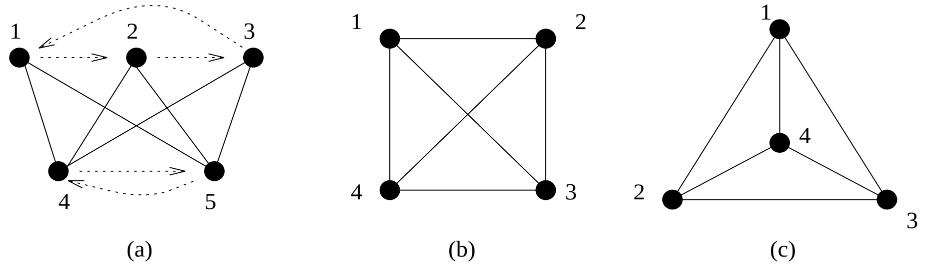

Figure 2: Examples

In this paper, we define a "geometric" automorphism, and a "geometric automorphism" group. Then we show that a symmetry of a graph layout corresponds to a geometric automorphism; and all symmetries of a graph layout correspond to a geometric automorphism group. Conversely, we prove that any geometric automorphism can be displayed as a symmetry of a layout; and all automorphisms in any given geometric automorphism group can be displayed simultaneously as symmetries by one graph layout. Thus, the representation of the symmetries of a graph layout as a geometric automorphism group is *complete*.

The earlier research from [14] also partially addressed the problem of characterization of symmetries, but concentrated only on a special class of graphs - *perfectly drawable* graphs. Our results as mentioned above extend the research from [14]. This is the first contribution of our paper.

Note that a spring algorithm A often randomly produces a drawing; and the possible drawings, which A can produce, correspond to the locally minimum energy configurations. The second and principal contribution of our paper is to provide some theoretical evidence of why spring algorithms display symmetries: we show that given a graph G and a spring algorithm A for drawing G, each geometric automorphism group of G can be displayed by A. This result holds for a wide variety of spring algorithms. To avoid the examination of various existing spring algorithms one by one, in this paper we model a general spring system, and undertake our investigation on the general spring system. Finally, we illustrate applications of our general theorem to several existing spring systems.

The rest of the paper is organized as follows. In section 2, we define a graph "drawing" (layout), geometric automorphism, geometric automorphism group and symmetry of a graph drawing; and show the completeness of our representation of the symmetries of a graph drawing in terms of a geometric automorphism group. Section 3 specifies our general spring system. In section 4, we show why spring algorithms display symmetry. Section 5 concludes the paper.

2 Symmetric Graph Drawings

In this Section, we represent symmetries of a graph drawing in terms of a geometric automorphism group of the graph; and then show the completeness of our representation. We start from defining a graph drawing.

2.1 Graphs and Graph Drawings

Suppose that $G = (V, E)$ is a graph. A *vertex sequence* $\mathcal{V}$ of $G = (V, E)$ consists of the vertices in V together with an integer subscript from 1 to $|V|$ for each vertex in V, and is represented by $(u_1, u_2, \ldots, u_{|V|})$. Sometimes, we denote V, E and $\mathcal{V}$ respectively by V_G, E_G and $\mathcal{V}_G$ to emphasize that they are from G.

In this paper, we discuss only "straight line" graph drawings in the two dimensional Euclidean space $\Re^2$. A *straight line drawing* $D(G)$ of G is uniquely determined by a mapping $D : V \rightarrow \Re^2$; the vertex v is placed at $D(v)$ and each edge $(u, v) \in E$ is represented by the line segment between $D(u)$ and $D(v)$. Here, with respect to a drawing $D(G)$, we use $D(\mathcal{V})$ to denote $(D(u_1), D(u_2), \ldots, D(u_{|V|}))$. A drawing $D(G)$ is *proper* if the mapping D on V is one-to-one.

2.2 Geometric Automorphisms

Let V be a finite set. A *permutation* on V is a one-to-one mapping from V to V. A non-empty set P of permutations on V is a *permutation group* if it is closed under composition and inversion.

The theory of permutation groups has a long history in mathematics (see [26]); here we review some of the relevant parts of this theory, and adapt them for the study at hand.

We denote the identity permutation by I, and the identity group $\{I\}$ also by I. If a permutation p acting on a set V has a *fixed element* $v \in V$, that is, $p(v) = v$, then p induces a permutation p_{-v} on $V - \{v\}$.

Given a permutation group P on V, a subset U of V is a *fixed block* of P if for every $p \in P$ and $u \in U$, $p(u) \in U$. A fixed block U of P is an *orbit* of P if any nonempty proper subset U_1 of U is not a fixed block of P. Note that if the size of a fixed block U is one, then the element in U is a fixed element of P.

A permutation group P is *semiregular* if each nonidentity permutation in P has no fixed element. It can be shown [26] that every orbit of a semiregular permutation group has the same size, and that the size of a semiregular permutation group is equal to the orbit size.

If $\{p_i : 1 \leq i \leq k\}$ are distinct permutations on V then the group *generated* by $\{p_1, p_2, \ldots, p_k\}$ is the smallest permutation group which contains $\{p_i : 1 \leq i \leq k\}$ and is denoted by $< p_1, p_2, \ldots, p_k >$. The p_i are called the *generators* of the group $< p_1, p_2, \ldots, p_k >$.

A permutation p on V is a *k-rotational* permutation if either $< p >$ or $< p_{-v} >$ (for some $v \in V$) is a semiregular permutation group, and $| < p > | = k > 1$. Here, the permutation group $< p_{-v} >$ acts on $V - \{v\}$. Note that a k-rotational permutation has at most one fixed element. A permutation p on V is a *reflectional* permutation if $p^2 = I$ and p is a nonidentity permutation. A permutation is a *geometric* permutation if it is either a reflectional permutation or a rotational permutation.

The concepts of reflectional and rotational permutation overlap: a reflectional permutation with at most one fixed element is a 2-rotational permutation, and *vice versa.*

A permutation group P on V is *geometric* if either 1) $P =< q >$ where q is a reflectional permutation; or 2) $P =< p >$ where p is a k-rotational permutation; or 3) $P =< p, q >$ where p is a k-rotational permutation and q is a reflectional permutation, and $< p > \cap < q >= I$, and $qp = p^{-1}q$. Note that in the third case P is also called *dihedral* in the usual terminology of group theory.

The following properties of a geometric group are useful later in the proofs of the other theorems.

Proposition 1 *Suppose that $< p, q >$ on V is geometric, where q is a reflectional permutation and p is a k-rotational permutation ($k \geq 2$). Then*

(a) each orbit of $< q >$ has either size 1 or size 2;

(b) each orbit of $< p >$ has size k, possibly except for one orbit of $< p >$ with size 1;

(c) a size k orbit U of $< p >$ can be represented by $\{u, p(u), \ldots, p^{k-1}(u)\}$ for an arbitrary $u \in U$;

(d) $p(v) = v$ implies that $q(v) = v$; and

(e) for each pair $\{u, v\}$ of vertices, if $q(u) = v$ then for $0 \leq i \leq k-1$, $qp^i(u) = p^{k-i}(v)$.

Proposition 1 is intuitively true if we think about a rotation and a reflection in $\Re^2$ together with their compositions, and can be immediately proved using elementary group theory. Lemmas 1 and 2 below follow immediately from Proposition 1.

Lemma 1 *Any nonidentity permutation p in a geometric permutation group is either a reflectional permutation or a rotational permutation. Consequently, p is a geometric permutation.*

Lemma 2 *Suppose that $< p, q >$ on V is geometric, where q is a reflectional permutation and p is a k-rotational permutation ($k \geq 2$). Then*

(a) q maps each orbit of $< p >$ into either itself or another orbit of $< p >$;

(b) an orbit U of $< p >$ contains one and only one fixed element of q if q maps U to itself and k is odd; and

(c) an orbit U of $< p >$ may contain either two fixed elements of q or none if k is even.

An *automorphism* of a graph $G = (V, E)$ is a permutation p on V such that if $(u, v) \in E$ then $(p(u), p(v)) \in E$. All automorphisms of a graph form the *automorphism group* of G [1]. An automorphism of G is a *geometric* automorphism if it is a geometric permutation on V; and thus it is called either a *reflectional automorphism* or a *k-rotational automorphism* accordingly. A subgroup P of the automorphism group of a graph G is called a *geometric* automorphism group if P is a geometric permutation group on V.

2.3 Symmetries and the Completeness of the Representation

A graph drawing can be thought of as a finite set of line segments and points in $\Re^2$. The symmetries of such a set are relatively simple. In this subsection, we characterize the symmetries of a proper graph drawing; and show that the representation of the symmetries of a graph drawing in terms of a geometric automorphism group is complete.

The results in this subsection are basically derived from [18] for pictures in $\Re^2$, and are adapted here for graph drawing purposes. Most of these results are quite intuitive, and can be immediately verified using elementary finite geometry and group theory. Thus, we do not detail every proof in this section, and provide only a brief derivation for most results. Interested readers may refer to [13] for detailed proofs.

A *transformation* on $\Re^2$ is a one-to-one mapping from $\Re^2$ onto $\Re^2$. An *isometry* on $\Re^2$ is a transformation on $\Re^2$ which preserves Euclidean distances. An isometry on $\Re^2$ preserves straight lines and angles. Consequently [18], an isometry in $\Re^2$ is either a *rotation* R, or a *reflection* L, or a *translation* T_X *by a vector* X on $\Re^2$, or a composition of them. The fixed point of R is called the *centre* of R, while the fixed line ζ of L is called the *axis* of L. An isometry which fixes a set Q of points in $\Re^2$ is a *symmetry* of Q ([18], pp. 28). The following Lemma limits the types of symmetries.

Lemma 3 *Suppose that Q is a nonempty and bounded set of points in $\Re^2$. Then each symmetry of Q is a reflection or a rotation.*

Proof: Suppose that α is a symmetry of Q. Since Q is nonempty and finite, α fixes the barycentre of Q. From Theorems 5.2, 5.3, 5.4, and 6.7 in [18], it follows that α is either a rotation or a reflection. $\Box$

It is straightforward to verify that the symmetries of a finite point set Q form a group under composition and inversion; this is the *symmetry group* of Q.

Given a graph $G = (V, E)$ and a proper drawing $D(G)$, the symmetry group of $D(V)$ is denoted by $S(V, D)$. A symmetry in $S(V, D)$ is either a *reflectional* symmetry or a *rotational* symmetry accordingly.

The restriction of a symmetry α in $S(V, D)$ to $D(V)$ generates a permutation p on V such that $p(u) = v$ if and only if $\alpha D(u) = D(v)$. Clearly, such a permutation p on V is uniquely determined by a given symmetry α in $S(V, D)$; and we say that α *induces* p. It is clear that the induced permutations on V by $S(V, D)$ form a permutation group $P(V, D)$. In general, we can verify that $P(V, D)$ is *isomorphic* to a subgroup of $S(V, D)$; but a permutation p in $P(V, D)$ may be induced by more than one symmetry in $S(V, D)$ unless there are at least three noncollinear points in $D(V)$. This implies that $P(V, D)$ is isomorphic to $S(V, D)$ if there are three noncollinear points in $D(V)$.

A "symmetry" of a drawing $D(G)$ first should be a symmetry of $D(V)$. However, to define a symmetry of $D(G)$ one has to specify some symmetric relationship among edges as well. This can be easily done for straight line drawings. A *symmetry* α of a proper drawing $D(G)$ is a symmetry of $D(V)$, such that for each pair of points $\{D(u), D(v)\}$, if $D(u)$ and $D(v)$ are connected by a line segment in $D(G)$, then $\alpha D(u)$ and $\alpha D(v)$ are also connected by a line segment in $D(G)$. The following proposition is immediate.

Proposition 2 *Given a proper drawing $D(G)$, a symmetry α of $D(V_G)$ is also a symmetry of $D(G)$ if and only if the induced permutation on V by α is a geometric automorphism of G.*

Similarly, the symmetries of a proper drawing $D(G)$ form a group $S(G, D)$, and the induced geometric automorphisms by the restriction of $S(G, D)$ to V_G form a group $P(G, D)$. The relationship between $S(G, D)$ and $P(G, D)$ is the same as that between $S(V, D)$ and $P(V, D)$. Further, by elementary permutation group theory and the definition of a geometric automorphism, the following Lemma is immediate.

Lemma 4 *Given any proper drawing $D(G)$, $P(G, D)$ is a geometric automorphism group of G.*

Up to this point, we have shown that any symmetry of a proper graph drawing $D(G)$ induces a geometric automorphism of G, and the symmetries of $D(G)$ form a group which induces a geometric automorphism group of G. Next, we consider the following problem: given a geometric automorphism group P of a graph G, is there a drawing D of G such that $P \subseteq P(G, D)$?

A proper drawing $D(G)$ of a graph G *displays* a geometric automorphism p of G if there is a symmetry α of $D(G)$ which induces p. A proper drawing $D(G)$ *displays* a geometric automorphism group P of a graph G if $D(G)$ displays every nonidentity element of P.

Proposition 3 *Suppose that P is a geometric automorphism group of a graph G. Then, there is a proper drawing $D(G)$ which displays P.*

According to Lemma 4, Proposition 3 holds if and only if the following lemma holds.

Lemma 5 *Suppose that P is a geometric automorphism group of a graph G. Then, there is a proper drawing $D(G)$ which displays the generators of P.*

Proof: According to the definition, P has three possible forms; and the most complicated form is that P is dihedral, that is $P =< p, q >$ where p is a k-rotational automorphism and q is a reflectional automorphism, $< p > \cap < q >= I$, and $qp = p^{-1}q$. We prove the lemma by providing a drawing algorithm for this form. The other two cases can be verified immediately by a slight modification of the drawing algorithm.

Orbits of $< p >$ with size k are drawn on concentric circles as follows, while the fixed element of $< p >$ (if it exists) is drawn at the center of these circles. Without loss of generality, we assume the centre is the origin $(0, 0)$.

Drawing Orbits:

For each orbit U of $< p >$, there are three cases:

case 1: q maps U to itself and there is no fixed element of q in U; or

case 2: q maps U to itself and there is (at least) one fixed element v_0 of q in U; or

case 3: q maps U to another orbit U_1.

For case 1, we can immediately verify, based on Proposition 1, that there is one vertex $u \in U$ such that $q(u) = p^{k-1}(u)$. Note that $U = \{p^i(u) : 0 \leq i \leq k-1\}$ where $p^0 = I$. Thus, we evenly space the vertices in U on a concentric circle such that for $0 \leq i \leq k-1$, $p^i(u)$ is successively placed according to counterclockwise direction, and the y-axis equally divides the arc between the location of u and the location of $p^{k-1}(u)$.

For case 2, note that $U = \{p^i(v_0) : 0 \leq i \leq k-1\}$. Again, we evenly space the vertices in U on a concentric circle such that for $0 \leq i \leq k-1$, $p^i(v_0)$ is successively placed according to counterclockwise direction, and v_0 is placed on y-axis.

For case 3, randomly choose a vertex $u \in U$, let $v = q(u)$. Note that $U = \{p^i(u) : 0 \leq i \leq k-1\}$ and $U_1 = \{p^i(v) : 0 \leq i \leq k-1\}$. We evenly space the vertices in U on a concentric circle C as well as evenly space the vertices in U_1 on C, such that:

- for $0 \leq i \leq k-1$, $p^i(u)$ is successively placed according to counterclockwise direction as well as $P^i(v)$, and
- the y-axis equally divides the arc between the location of u and the location of v.

From Proposition 1, Lemma 1, and Lemma 2, it immediately follows that the drawing displays p and q and the y-axis is the axis of the reflectional symmetry. □

Recall the two examples in Section 1 (see Figure 2). The automorphism p with respect to Figure 2(a) is not a geometric automorphism according to the definition; while the two automorphisms p_1 and p_2 with respect to Figures 2(b) and 2(c) cannot coexist in any geometric automorphism group.

3 A General Spring System

The basic idea behind spring algorithms for drawing graphs is to simulate a "balanced state" of a mechanical system. A spring algorithm consists of two parts [2, 6, 8, 12, 23]: 1) A "spring system" - a mechanical model for graphs; and 2) A technique for finding a minimal energy state of the spring system that gives the drawing of the graph.

In the first part, the vertices of a graph are replaced by steel rings, and a set of pairs of vertices are assembled springs. There are two kinds of steel rings: *movable* rings which can be moved in $\Re^2$ from one position to another position; and *fixed rings* which are nailed down in certain positions. Consequently, the vertices to be replaced by fixed rings are called *fixed vertices*. A *state* of the spring system consists of positions for each ring while the positions for fixed rings are retained.

In the second part, the algorithmic nature of a spring algorithm is implemented: an algorithm is developed for finding the minimal energy state of the spring system. Normally, it is quite difficult to find a globally minimal energy state. Instead, a *balanced* state is usually sought [2, 6, 8, 12, 23], where at each movable ring, the total forces of the incident springs are zero.

A number of spring systems have been proposed [2, 4, 6, 8, 12, 13, 23]; some of these are described in Section 3.3. Most of the existing spring systems [2, 6, 8, 24] do not follow Hooke's Law strictly. Instead, they applied various generalizations which retain the essence of Hooke's Law.

3.1 The Generalized Hooke's Law

A spring has a *natural length* l and the current length d between its two ends. Hooke's Law [20] specifies the relationship among the spring length and energy and forces as follows. 1) The spring has zero force and zero energy if $l = d$. 2) The spring attracts the two ends if $d > l$, and repels two the ends if $d < l$. 3) Both of the attractive and repulsive forces are proportional to $|d - l|$. 4) The energy is the integral of the forces with respect to $|d - l|$.

In our general spring model, we retain the above properties 1), 2), and 4) but give a generalization to the third: the force function of a generalized spring is a *continuous* and *increasing* function with respect to $|d - l|$ and can be viewed as the derivative of the energy function. Further, we allow an infinite natural length. These can be defined as follows.

A generalized spring is represented as (f, l), where l is the natural length - a non-negative real number or infinity, and the energy function f is a mapping from $\Re$ to $\Re^+$, where $\Re^+$ is the set of nonnegative real numbers, such that:

- if $f \equiv 0$, then $l = 0$;
- if $f \not\equiv 0$ and $l \neq \infty$, then $f(0) = 0$, f is *differentiable* [21], the derivative f' is continuous and nondecreasing, $f'(0) = 0$, and $f'(x) \neq 0$ if $x \neq 0$;
- if $l = \infty$, then f is defined on $\Re^+$, $f(0) = 0$, f is differentiable, and $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to\infty} f(x) = \infty$, $x^2 f'(x)$ is nondecreasing and continuous for $x > 0$, and $\lim_{x\to 0} x^2 f'(x) = 0$.

Suppose that $D(G)$ is a drawing of a graph $G = (V, E)$, and $u, v \in V$. We denote the Euclidean distance between $D(u)$ and $D(v)$ by $|D(u) - D(v)|$, and if $D(u) \neq D(v)$, then we denote the unit vector in the direction from $D(v)$ to $D(u)$ by $T_D(u, v)$.

Suppose that $D(u)$ and $D(v)$ are two ends of a spring (f, l). The difference h_{uv} between $|D(u) - D(v)|$ and l is defined as:

$$h_{uv} = \begin{cases} |D(u) - D(v)| - l & \text{if } l \neq \infty, \text{ or} \\ \frac{1}{|D(u)-D(v)|} & \text{if } l = \infty; \end{cases} \tag{1}$$

The generalized Hooke's Law specifies the energy and force of a spring (f, l) as follows. The energy of the spring is $f(h_{uv})$ if either $D(u) \neq D(v)$ or $l \neq \infty$. Otherwise ($D(u) = D(v)$ and $l = \infty$) the energy is infinite. The force on u is either:

- $f'(|D(u) - D(v)| - l)T_D(v, u)$ if $D(u) \neq D(v)$ and $l \neq \infty$; or
- $f'(\frac{1}{|D(u)-D(v)|})\frac{1}{|D(u)-D(v)|^2}T_D(u, v)$ if $D(u) \neq D(v)$ and $l = \infty$; or
- 0 if $D(u) = D(v)$ and $l = 0$; or
- undefined if $D(u) = D(v)$ and $l > 0$.

3.2 A General Spring System

A *spring assignment* $\{\mathcal{F}, \mathcal{L}, G\}$ on a graph G consists of a spring (f_{uv}, l_{uv}) for each pair u, v of distinct vertices. That is $\{\mathcal{F}, \mathcal{L}, G\} = \{(f_{uv}, l_{uv}) : u, v \in V_G \ \&\ u \neq v\}$.

To display a graph structure, the following restrictions are added while designing a spring system. Given a graph $G = (V, E)$ and a subset Ω (the set of fixed vertices) of V, a spring assignment *respects* G and Ω if:

- $f_{u,v} \equiv 0$ when $u, v \in \Omega$;
- $f_{uv} = f_{ab}$ and $l_{uv} = l_{ab}$ when the graph-theoretic distance between a and b is the same as that between u and v, $(V - \Omega) \cap \{u, v\} \neq \emptyset$, and $(V - \Omega) \cap \{a, b\} \neq \emptyset$;
- $f_{uv} \not\equiv 0$ and $l_{uv} \neq \infty$ when $(u, v) \in E$ and $(V - \Omega) \cap \{u, v\} \neq \emptyset$.

A *general spring system* $\{\mathcal{F}, \mathcal{L}, G, D_0(\Omega)\}$ of a graph G, where $\Omega \in V_G$, consists of a proper drawing $D_0(\Omega)$ of Ω and a spring assignment $\{\mathcal{F}, \mathcal{L}, G\}$ respecting G and Ω. Note that in a general spring system the *fixed vertex set* Ω may be empty.

A *state* of $\{\mathcal{F}, \mathcal{L}, G, D_0(\Omega)\}$ is a drawing $D(G)$ such that $D(G)$ restricted to Ω is $D_0(\Omega)$. A state $D(G)$ is a *finite energy* state if for every pair $\{u, v\}$ of distinct vertices, either $D(u) \neq D(v)$ or $l_{uv} \neq \infty$.

The *total spring energy* at a finite energy state $D(G)$ is finite, and can be represented as:

$$\sum_{\{u,v\} \subset V, u \neq v} f_{uv}(h_{uv}), \tag{2}$$

where h_{uv} is as defined in (1). The *total spring energy* at a non-finite energy is infinity.

Suppose that G has n vertices. As mentioned before, a graph drawing $D(G)$ is uniquely determined by $D(\mathcal{V}_G)$. If each $D(\mathcal{V}_G)$ is viewed as a point in $2n$-dimensional Euclidean space $\Re^{2n}$, then the total spring energy (2) at finite energy states $D(G)$ can be also viewed as a non-negative real valued function $F_{\mathcal{V}_G}$ on $\Re^{2n}$ and is denoted by $F_{\mathcal{V}_G}(D(\mathcal{V}_G))$. We call $F_{\mathcal{V}_G}$ a *total spring energy function*.

A state $D(G)$ of a general spring system $\{\mathcal{F}, \mathcal{L}, G, D_0(\Omega)\}$ is *stable* if $D(G)$ is a finite energy state, $F_{\mathcal{V}_G}$ is differentiable at $D(\mathcal{V}_G)$, and for each vertex $u \in V - \Omega$, the partial derivatives of $F_{\mathcal{V}_G}$ respectively with respect to the x and y coordinates of $D(u)$ are both zero at $D(\mathcal{V}_G)$.

The following Lemma is immediate; it specifies the conditions under which $F_{\mathcal{V}_G}$ is differentiable at $D(\mathcal{V}_G)$.

Lemma 6 *Suppose that $D(G)$ is a finite energy state of a $\{\mathcal{F}, \mathcal{L}, G, D_0(\Omega)\}$. Then, $F_{\mathcal{V}_G}$ is differentiable at $D(\mathcal{V}_G)$ if and only if for every pair u, v of distinct vertices with $D(u) = D(v)$, $l_{uv} = 0$.*

Intuitively, a stable state of a general spring system is a balanced state according to the generalized Hooke's Law. This intuition can be formally verified by elementary calculus.

Lemma 7 *If a state $D(G)$ of $\{\mathcal{F}, \mathcal{L}, G, D_0(\Omega)\}$ is stable then there are no undefined forces in the state, and for every vertex $u \in V - \Omega$, the summation of all spring forces on u is zero.*

3.3 Generality of Our Spring System

The spring systems in [2, 6, 8, 12, 24] are instances of the proposed general spring system. Below, we show how these systems can be described using our general spring system model.

Tutte's System [24]. In this spring system, the fixed set Ω is the vertex set of a *face* of the graph [13, 24], and $D_0(\Omega)$ is a convex polygon. For each pair u, v of distinct vertices, if $(u, v) \in E$ and one of u and v is not in Ω then $f_{uv}(x) = x^2$ and $l_{uv} = 0$; otherwise $f_{uv} \equiv 0$ and $l_{uv} = 0$.

Becker and Hotz' System [2]. The system is a generalization of Tutte's system. Instead of using a quadratic function for each f_{uv}, a continuous and strictly increasing [21] function g with $g(0) = 0$ is given. As with Tutte's system, $l_{uv} = 0$ for all pairs u, v.

Energy Version of Eades' System [6]. Here $\Omega = \emptyset$, and the system chooses the spring energy functions and natural lengths as follows

- $f_{uv} = \int_0^x C_1 \log(\frac{y}{l} + 1) dy$ and $l_{uv} = l$ if $(u, v) \in E$, and
- $f_{uv} = C_2 x$ and $l_{uv} = \infty$ if $(u, v) \notin E$,

where l, C_1, and C_2 are positive constants.

Kamada and Kawai's System [12]. Here $\Omega = \emptyset$, and $f_{uv}(x) = C_1 \frac{x^2}{\sigma(u,v)^2}$ with $l_{uv} = C_2 \sigma(u, v)$ where $\sigma(u, v)$ represents the graph-theoretic distance between u and v, and C_1 and C_2 are positive constants.

Clearly, Tutte's system, Kamada and Kawai's system, and the energy version of Eades' system are instances of our general spring system. Furthermore, if we add the following restrictions to the function g in the system of Becker and Hotz.

> g is differentiable, g' is continuous and nondecreasing with $g'(0) = 0$ and $g'(x) \neq 0$ if $x \neq 0$.

Then Becker and Hotz' system is also an instance of our general spring systems. Although the spring system in [8] - a generalization of Eades spring system [6] - is beyond our general spring system coverage, it can be viewed as a combination of two instances of the general spring system: for a given graph G, the spring system in [8] is composed of

1. an instance of the general spring model on G, and

2. an instance of the general spring system on the *complement* [1] of G

There are some spring systems which are not covered by the general spring system; for example, the magnetic spring system [22] of Misue and the fast system [25] of Tunkelang. However, these systems are not specifically designed for producing symmetric drawings.

4 Display of Symmetries

In this section, we present the major result, Theorem 1, of this paper, that for each geometric automorphism group P of a graph G and any spring system A within our general spring system model, there is a stable state of A which can display P.

All existing spring algorithms [2, 6, 8, 12, 24] non-deterministically produce a balanced state for their spring systems. From Lemma 7 and Theorem 1, it follows that an existing spring algorithm can display any geometric automorphism group.

Note that some of the spring systems [2, 24] have non-empty fixed vertex sets Ω. To ensure that a produced graph drawing is symmetric, we need the following restrictions on a given $D_0(\Omega)$ and a given geometric automorphism group P. When $\Omega \neq \emptyset$, a geometric automorphism group P of G is *compatible* with $D_0(G)$ if 1) P restricted to Ω is either a geometric automorphism group of G_Ω or an identity group where G_Ω is the *induced* subgraph on Ω from G; 2) the straight line drawing $D_0(G_\Omega)$, determined by $D_0(\Omega)$, displays the restriction of P to G_Ω if the restriction is not an identity group; and 3) the points in $D_0(\Omega)$ are located on one line if P restricted to G_Ω is the identity group.

Note that we assume that a geometric automorphism group P is always compatible with $D_0(\emptyset)$. The main result is as follows.

Theorem 1 *In a spring system $\{\mathcal{F}, \mathcal{L}, G, D_0(\Omega)\}$, suppose that P is a geometric automorphism group of G and is compatible with $D_0(\Omega)$. Then, there is a drawing $D(G)$ which is a stable state, and displays P.*

The Theorem is trivially true if $\Omega = V$, since P is compatible with $D_0(\Omega)$. However, the proof of this Theorem is quite long when $\Omega \subset V$; and thus it is presented in the next several subsections.

The scope of Theorem 1 includes a case where a stable state is an improper drawing. In practice, a good graph drawing algorithm should at least output a proper drawing. To retain the generality of the result for theoretical interest, we still include cases of improper drawings in the Theorem. Therefore, we will extend the concepts of symmetry display from proper drawings to improper drawings in Section 4.1 below.

Lemma 8 *Suppose that in a spring system $\{\mathcal{F}, \mathcal{L}, G, D_0(\Omega)\}$, $f_{uv} \not\equiv 0$ and $l_{uv} > 0$ for each pair u, v of distinct vertices where one of u and v is not in Ω. Further, P is a geometric automorphism of G, and is compatible with $D_0(\Omega)$. Then, each stable state is proper.*

Proof: The lemma immediately follows from Lemma 6 and the definition of a stable state. □

Clearly, the spring systems in [2, 6, 12, 24] are within the scope of Lemma 8. Thus the spring algorithms in [2, 6, 12, 24] seeking stable states for their spring systems always output proper drawings. Consequently, these spring algorithms can non-deterministically produce a proper drawing to display any given geometric automorphism group. This gives the practical grounding of Theorem 1.

In the next three subsections, we will prove Theorem 1. The basic idea is very simple. Firstly, we can show, as with Proposition 3, that there is a proper drawing $D(G)$ whose restriction to Ω is $D_0(\Omega)$ and which displays P (Lemma 11). Secondly, we can show that if $D(G)$ is unstable, then $D(G)$ "converges" to a stable state after a sequence of modifications that preserve all symmetries and leave the drawing of Ω unchanged. More specifically, our proof proceeds as follows.

Step 1: We prove that if a drawing $D(G)$ is a finite energy state but unstable, then there are at least one vertex u in $V-\Omega$ together with a 2-dimensional "direction" such that moving the vertex u along the direction reduces the total spring energy. (Theorems 2 and 3 in Section 4.2.)

Step 2: Using the results in Step 1 we can prove that if a drawing $D(G)$ is a finite energy state but unstable, then there is a drawing $\bar{D}(G)$ (obtained by a symmetric modification of $D(G)$) with smaller total energy which retains all symmetries of $D(G)$. (Lemmas 15 and 16 in Section 4.3.)

Step 3: Define a "closed" set $S_{G,P,\Omega}$ consisting of layouts $\tilde{D}(G)$ such that $\tilde{D}(G)$ is a finite state and displays P. By choosing the "greatest lower bound" ([21], p. 93) c of $\{F_{\mathcal{V}_G}(\tilde{D}(\mathcal{V}_G)) : \tilde{D}(\mathcal{V}_G) \in S_{G,P,\Omega}\}$ and using the results in the Step 2, we can show that there is a drawing $D(G)$ with the property that $D(G)$ is in $S_{G,P,\Omega}$, $F_{\mathcal{V}_G}(D(\mathcal{V}_G)) = c$ and $D(G)$ is a stable state. This concludes our proof.

4.1 A Note on Display of Graph Symmetries

We shall define a symmetry for improper drawings of graph in the same way as that for proper drawings. In a proper drawing there is almost an one-to-one mapping between a symmetry and a geometric automorphism except when all vertices are allocated on a line. Improper drawings are different. In an improper drawing, one can imagine the extreme case where all the vertices of a graph are located at one point. Thus, in improper drawings we should more explicitly specify the mapping between a symmetry and a geometric automorphism and then define the concept of display of a geometric automorphism. We have developed the following definitions that extend the given definitions in Section 2.3 for proper drawings.

Suppose that $D(G)$ is a drawing (possibly improper) of an undirected graph $G=(V,E)$, q is a reflectional automorphism, and p is a k-rotational automorphism. Then, a symmetry β of $D(G)$ *induces* q if

RE1: β is the reflectional symmetry of $D(G)$ with axis ζ;

RE2: for every vertex $u \in V$, ζ contains the barycentre of $D(u)$ and $D(q(u))$; and

RE3: for every vertex $u \in V$, if $D(u) \neq D(q(u))$ then ζ is perpendicular to the line segment from $D(u)$ to $D(q(u))$.

A symmetry α *induces* p if

RO1: α is the rotational symmetry - a rotation by $\frac{2\pi}{k}$ with the center O;

RO2: $D(v) = O$ when v is the fixed element (vertex) of $< p >$; and

RO3: for every orbit $\{u, p(u), ...p^{k-1}(u)\}$ of $< p >$, either

- all vertices of the orbit are mapped (by D) evenly on a circle with the centre X such that for $1 \leq i \leq k$, D maps $p^i(u)$ after $p^{i-1}(u)$ in the counterclockwise direction, or
- all vertices of the orbit are mapped to O by D.

We can verify the following Lemma immediately.

Lemma 9 *Suppose that α is a symmetry of $D(G)$ and induces a geometric automorphism p of G. Then, for each pair of vertices u and v, $p(u) = v$ implies that $\alpha D(u) = D(v)$.*

As with proper drawings, $D(G)$ *displays* a geometric automorphism p if there is a symmetry which induces p; and $D(G)$ *displays* a geometric automorphism group P if $D(G)$ displays every nonidentity element of P. Immediately, we can verify that the above definitions are equivalent to the corresponding definitions given in Section 2.3, when restricted to proper drawings.

Further, it is clear that Lemma 5 also holds for improper drawings.

Lemma 10 *Suppose that $D(G)$ is a drawing (possibly improper), and P is a geometric automorphism group of G. Then, the drawing $D(G)$ displays P if and only if $D(G)$ displays the generators of P.*

The following Lemma can be proved along the lines of the proof of Proposition 3.

Lemma 11 *Suppose that a spring system $\{\mathcal{F}, \mathcal{L}, G, D_0(\Omega)\}$ and a geometric automorphism group P of G are given such that P is compatible with $D_0(\Omega)$. Then there is a proper drawing $D(G)$ such that $D(G)$ is a state of the spring system, and displays P.*

4.2 Energy of Unstable States Reducible

In this subsection, we detail Step 1 in the proof of Theorem 1.

In the rest of this paper, sometimes we use a vector d in $\Re^{2n}$ to represent a set of n vectors $\{\delta_i = (w_{ix}, w_{iy}) : 1 \leq i \leq n\}$ in $\Re^2$. Here, d is represented as either $(\delta_1, \delta_2, ..., \delta_n)$ or $(w_{1x},\ w_{1y}\ , ...,\ w_{nx},\ w_{ny})$.

The value of $\frac{\partial f(x_1,...,x_n)}{\partial x_i}$ at the point $(y_1, ..., y_n)$ is denoted by $\left(\frac{\partial f(x_1,...,x_n)}{\partial x_i}\right)_{(y_1,...,y_n)}$. A real valued function g on $\Re^n$ *has a directional derivative in direction d at a point X_0* if

$$\lim_{t \to +0} \frac{g(X_0 + td) - g(X_0)}{t}$$

is finite. The value of the limit is called the *directional derivative of g in direction d at the point X_0*, and is denoted by $\left(\frac{\partial g(X)}{\partial d}\right)_{X_0}$.

By elementary calculus, one can immediately verify the following two Lemmas.

Lemma 12 *For any finite energy state $D(G)$ of a spring system $\{\mathcal{F}, \mathcal{L}, G, D_0(\Omega)\}$, the total spring energy function $F_{\mathcal{V}_G}$ always has a directional derivative at $D(\mathcal{V}_G)$ with respect to any direction d.*

Lemma 13 *Suppose that $D(G)$ is a finite energy state of a spring system $\{\mathcal{F}, \mathcal{L}, G, D_0(\Omega)\}$ where G has n vertices and $\{d_i : 1 \le i \le k\}$ is a set of directions in $\Re^{2n}$. Further suppose that the total spring energy function $F_{\mathcal{V}_G}$ is differentiable at $D(\mathcal{V}_G)$. Then:*

1. *If $d = \sum_{i=1}^{k} a_i d_i$ is a direction in $\Re^{2n}$ where a_i $(1 \le i \le k$ $)$ are real numbers, then*

$$\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d}\right)_{D(\mathcal{V}_G)} = \sum_{i=1}^{k} a_i \left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d_i}\right)_{D(\mathcal{V}_G)};$$

and

2. *if $d = (w_1,\ w_2, ...,\ w_{2n})$, then*

$$\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d}\right)_{D(\mathcal{V}_G)} = \sum_{i=1}^{n} w_{2i-1} \left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial x_i}\right)_{D(\mathcal{V}_G)} + \sum_{i=1}^{n} w_{2i} \left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial y_i}\right)_{D(\mathcal{V}_G)}.$$

Intuitively, if a finite energy state D of a spring system is unstable, then it is not locally minimal; that is, we can find some vertices u_i $(1 \le i \le k)$ and their associated 2-dimensional directions δ_i $(1 \le i \le k)$ such that moving each u_i in direction δ_i $(1 \le i \le k)$ reduces the total spring energy. There are two kinds of unstable finite energy states: either differentiable or non-differentiable. Theorems 2 and 3 confirm the intuition with respect to these two cases. From Lemma 6, in a non-differentiable finite energy state, there must be some pair u, v of vertices such that $D(u) = D(v)$ and $l_{uv} > 0$.

Theorem 2 *Suppose that a finite energy state $D(G)$ of a spring system $\{\mathcal{F}, \mathcal{L}, G, D_0(\Omega)\}$ is unstable where G has n vertices, and the total spring energy function $F_{\mathcal{V}_G}$ is differentiable at $D(\mathcal{V}_G)$. Then there exists a vertex $u_i \in V - \Omega$ and a corresponding 2-dimensional direction (x, y) such that $\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d}\right)_{D(\mathcal{V}_G)} < 0$ where $d = (\delta_1,\ \delta_2,\ ...,\ \delta_n)$ is made from:*

$$\delta_j = \begin{cases} (0,0) & j \ne i \\ (x,y) & j = i. \end{cases}$$

Proof: Let $X = (x_1, y_1, ..., x_n, y_n)$. Since D is unstable, the definition of unstability implies that there is a $u_i \in V - \Omega$ such that

$$Q = \left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial x_i}\right)^2_{D(\mathcal{V}_G)} + \left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial y_i}\right)^2_{D(\mathcal{V}_G)} > 0.$$

We choose (x, y) as:

$$x = -\frac{\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial x_i}\right)_{D(\mathcal{V}_G)}}{Q^{\frac{1}{2}}}, \text{ and } y = -\frac{\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial y_i}\right)_{D(\mathcal{V}_G)}}{Q^{\frac{1}{2}}}.$$

According to Lemma 13, we can conclude that

$$\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d}\right)_{D(\mathcal{V}_G)} = -Q^{\frac{1}{2}} < 0.$$

□

Theorem 3 *Suppose that $\mathcal{V}_G$ is represented as $\{u_i : 1 \le i \le n\}$; and with respect to a finite energy state $D(G)$ of a spring system $\{\mathcal{F}, \mathcal{L}, G, D_0(\Omega)\}$, there are k pairs S_k of vertices in $\mathcal{V}_G$, where*

$$S_k = \{\{u_j, u_{i_j}\} : 1 \le j \le k \ \&\ u_j \in V - \Omega\},$$

such that for each pair of u_j and u_{i_j}, $l_{u_j u_{i_j}} > 0$ and $D(u_{i_j}) = D(u_j)$. Further, a $2n$-dimensional direction $d = (\delta_1,\ \delta_2,\ ...,\ \delta_n)$ is given, such that there is at least one pair of δ_z and δ_{i_z} with the property that $\delta_z \ne \delta_{i_z}$, and the corresponding $\{u_z, u_{i_z}\}$ is in S_k. Then either

$$\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d}\right)_{D(\mathcal{V}_G)} < 0, \ \ or \ \left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial (-d)}\right)_{D(\mathcal{V}_G)} < 0.$$

Proof: Note that $D(G)$ is a finite state of the spring system. Thus, from the definition, there are no pairs of vertices u and v such that $D(u) = D(v)$ and $l_{uv} = \infty$; that is, each pair $\{u, v\}$ of distinct vertices is in one of the following four sets:

- $U_1 = \{\{u, v\} :\ u \ne v,\ D(u) = D(v),\ l_{uv} > 0, l_{uv} \ne \infty\}$,
- $U_2 = \{\{u, v\} :\ u \ne v,\ D(u) = D(v),\ l_{uv} = 0\}$,
- $U_3 = \{\{u, v\} :\ u \ne v,\ D(u) \ne D(v),\ l_{uv} \ne \infty\}$,
- $U_4 = \{\{u, v\} :\ u \ne v,\ D(u) \ne D(v),\ l_{uv} = \infty\}$.

Let:

- $a = \sum_{\{u_i, u_j\} \in U_1} f'_{u_i u_j}(-l_{u_i u_j}) |\delta_i - \delta_j| \ + \ \sum_{\{u_i, u_j\} \in U_2} f'_{u_i u_j}(0) |\delta_i - \delta_j|$,
- $c = \sum_{\{u_i, u_j\} \in U_3} f'_{u_i u_j}(h_{u_i u_j})[(\delta_i - \delta_j) T_D(u_j, u_i)]$,
- $d = \ \sum_{\{u_i, u_j\} \in U_4} f'_{u_i u_j}(h_{u_i u_j}) h^2_{u_i u_j}[(\delta_i - \delta_j) T_D(u_i, u_j)]$.

The following is immediate:

$$\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d}\right)_{D(\mathcal{V}_G)} = a + c + d, \text{ and } \left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial (-d)}\right)_{D(\mathcal{V}_G)} = a - c - d.$$

Note that f_{uv} is not identically zero when $l_{uv} > 0$; $f'_{uv}(x) < 0$ for $x < 0$ when l_{uv} is a positive real number; and $f'_{uv}(0) = 0$. Further, in U_1 there is a pair $\{u_{i_z}, u_z\}$ of vertices such that $\delta_{i_z} \ne \delta_z$. Thus, $f'_{u_{i_z} u_z}(-l_{u_{i_z} u_z}) |\delta_{i_z} - \delta_z| < 0$. As $f'_{u_i u_j}(-l_{u_i u_j}) |\delta_i - \delta_j| \le 0$ for all pairs $\{u_i, u_j\} \in U_1$, we can conclude

$$\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d}\right)_{D(\mathcal{V}_G)} + \left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial (-d)}\right)_{D(\mathcal{V}_G)} = 2a < 0.$$

□

4.3 Symmetric Movements Reducing Energy

In this Section, we detail Step 2 in the proof of Theorem 1.

Lemma 14 *Suppose that $D(G)$ is a finite energy state of a spring system $\{\mathcal{F}, \mathcal{L}, G, D_0(\Omega)\}$ and displays a geometric automorphism p of G, that is, there is a symmetry α of $D(G)$ which induces p. Then for a pair $\{u_{i_0}, u_{j_0}\}$ (i_0 might be equal to j_0) of vertices such that $p(u_{i_0}) = u_{j_0}$, we have that*

$$\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d}\right)_{D(\mathcal{V}_G)} = \left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d'}\right)_{D(\mathcal{V}_G)},$$

where $d = (\delta_1,\ \delta_2,\ \ldots.\delta_n)$ and $d' = (\delta_1^{'},\ \delta_2^{'},\ \ldots,\ \delta_n^{'})$, such that

$$\delta_i = \begin{cases} (0,0) & \text{if } i \neq i_0 \\ (x,y) & i = i_0 \ \ (\ (x,y) \text{ is a direction in } \Re^2\ ), \end{cases}$$

and

$$\delta_j^{'} = \begin{cases} (0,0) & \text{if } j \neq j_0 \\ \alpha(x,y) & j = j_0. \end{cases}$$

Proof: Clearly, both d and d' are unit vectors and thus can be viewed as directions. By Lemma 12, $F_{\mathcal{V}_G}$ has a directional derivative at $D(\mathcal{V}_G)$ in any direction.

Clearly, we need only to prove that $F_{\mathcal{V}_G}(D(\mathcal{V}_G) + td) = F_{\mathcal{V}_G}(D(\mathcal{V}_G) + td')$ for any $t > 0$; and thus we need only to verify the following equations (3) and (4).

$$\begin{aligned} &\sum_{v \neq u_{i_0}, l_{u_{i_0}v} \neq \infty} f_{u_{i_0}v}(|D(u_{i_0}) + t\delta_{i_0} - D(v)| - l_{u_{i_0}v}) \\ = &\sum_{w \neq u_{j_0}, l_{u_{j_0}w} \neq \infty} f_{u_{j_0}w}(|D(u_{j_0}) + t\alpha\delta_{i_0} - D(w)| - l_{u_{j_0}w}), \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\sum_{v \neq u_{i_0}, l_{u_{i_0}v} = \infty} f_{u_{i_0}v}\left(\frac{1}{|D(u_{i_0}) + t\delta_{i_0} - D(v)|}\right) \\ = &\sum_{w \neq u_{j_0}, l_{u_{j_0}w} = \infty} f_{u_{j_0}w}\left(\frac{1}{|D(u_{j_0}) + t\alpha\delta_{i_0} - D(w)|}\right). \end{aligned} \tag{4}$$

Note that p is an automorphism of G, and also a one to one mapping on V_G. Thus, the graph theoretic distance between u_{i_0} and v is equal to the graph theoretic distance between $p(u_{i_0})$ and $p(v)$ for any vertex v. Hence, according to the definition of a general spring system, we have that $f_{u_{i_0}v} = f_{p(u_{i_0})p(v)}$ and $l_{u_{i_0}v} = l_{p(u_{i_0})p(v)}$. Since α is a symmetry, it preserves distances in $\Re^2$. Further, $D(p(v)) = \alpha D(v)$ for each vertex v by Lemma 9. Let $w = p(v)$ for every vertex v; we have

$$\begin{aligned}
&\sum_{w \neq u_{j_0}, l_{u_{j_0}w} \neq \infty} f_{u_{j_0}w}(|D(u_{j_0}) + \alpha t\delta_{i_0} - D(w)| - l_{u_{j_0}w}) \\
= &\sum_{v \neq u_{i_0}, l_{u_{i_0}v} \neq \infty} f_{p(u_{i_0})p(v)}(|\alpha(D(u_{i_0}) + t\delta_{i_0} - D(v))| - l_{p(u_{i_0})p(v)}) \\
= &\sum_{v \neq u_{i_0}, l_{u_{i_0}v} \neq \infty} f_{u_{i_0}v}(|D(u_{i_0}) + t\delta_{i_0} - D(v)| - l_{u_{i_0}v})
\end{aligned}$$

Similarly, we can prove (4). □

Lemma 15 *Suppose that $D(G)$ is an unstable and finite energy state of a $\{\mathcal{F}, \mathcal{L}, G, D_0(\Omega)\}$ and $D(G)$ displays P, where G has n vertices and P is a geometric automorphism group of G compatible with $D_0(\Omega)$. Further, suppose that the total spring energy function $F_{\mathcal{V}_G}$ is differentiable at $D(\mathcal{V}_G)$. Then, there is a drawing $\bar{D}(G)$ such that $\bar{D}(\mathcal{V}_G) = D(\mathcal{V}_G) + \epsilon\bar{d}$ where $\bar{d}$ is a direction in $\Re^{2n}$ and $\epsilon > 0$; $F_{\mathcal{V}_G}(\bar{D}(\mathcal{V}_G)) < F_{\mathcal{V}_G}(D(\mathcal{V}_G))$; and $\bar{D}(G)$ displays P.*

Proof: The basic idea of this proof is to construct such a direction $\bar{d}$ based on Theorem 2 and Lemma 14. This is conducted with respect to the three types of group P:

Case 1: $P = < q >$ where q is a reflectional automorphism.

Case 2: $P = < p >$ where p is a k-rotational ($k \geq 2$) automorphism.

Case 3: $P = < p, q >$ such that:

(1) q is a reflectional automorphism and a p is k-rotational automorphism ($k \geq 2$), and

(2) $< p > \cap < q > = I$, and

(3) $qp = p^{-1}q$.

We assume that in both Case 2 and Case 3, α is a k-rotational symmetry of $D(G)$ inducing p; and that in both Case 1 and Case 3, β is a reflectional symmetry of $D(G)$ inducing q. Without loss of generality, we can assume that the barycenter of $\{D(u) : u \in V_G\}$ is the origin $(0, 0)$. Consequently, α can be represented by a matrix

$$\alpha = R\left(\frac{2\pi}{k}\right) = \begin{pmatrix} \cos(\frac{2\pi}{k}) & -\sin(\frac{2\pi}{k}) \\ \sin(\frac{2\pi}{k}) & \cos(\frac{2\pi}{k}) \end{pmatrix}, \tag{5}$$

and β can be represented by another matrix

$$\beta = L(\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}, \tag{6}$$

where θ is the angle from the x-axis to the axis of β. Using the above matrix representations, for each point $s = (x, y) \in \Re^2$ we have

$$\alpha s = R\left(\frac{2\pi}{k}\right)\begin{pmatrix} x \\ y \end{pmatrix},$$

and

$$\beta s = L(\theta)\begin{pmatrix} x \\ y \end{pmatrix}.$$

As $D(G)$ is unstable, Theorem 2 says that there is a vertex $u_{i_0} \in V - \Omega$ and a corresponding direction $d = (\delta_1,\ \delta_2,\ ...,\ \delta_n)$ in $\Re^{2n}$, where $\delta_j = (0,0)$ $(j \neq i_0)$ and δ_{i_0} is a direction in $\Re^2$, such that

$$\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d}\right)_{D(\mathcal{V}_G)} < 0.$$

Without loss of generality, we can assume that $\mathcal{V}_G$ gives such a vertex ordering that $i_0 = 1$. Hence the direction d is $(\delta_1,\ (0,0),\ ...,\ (0,0))$.

Suppose that $\bar{d}$ is represented by $(\bar{\delta}_1,\ \bar{\delta}_2,\ ...,\ \bar{\delta}_n)$, where for $1 \leq i \leq n$, $\bar{\delta}_i$ is either a 2-dimensional direction or zero vector. Next we show how to construct $\bar{d}$ with respect to u_1, d, and the three types of P.

Case 1:

Let $u_i = q(u_1)$ and $d' = (\delta_1',\ \delta_2',\ ...,\ \delta_n')$ be such a direction that $\delta_j' = (0,0)$ if $j \neq i$, and $\delta_i' = \beta\delta_1$. From Lemma 14, it follows that

$$\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d'}\right)_{D(\mathcal{V}_G)} = \left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d}\right)_{D(\mathcal{V}_G)} < 0.$$

From Lemma 13(1), it follows that $d + d' \neq 0$. Let $\bar{d} = (d+d')/|d+d'|$. From Lemma 13, it follows that

$$\left(\frac{\partial F_(X)}{\partial \bar{d}}\right)_{D(\mathcal{V}_G)} = \frac{\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d}\right)_{D(\mathcal{V}_G)} + \left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d'}\right)_{D(\mathcal{V}_G)}}{|d+d'|} < 0.$$

Thus there is an $\epsilon > 0$ such that $F_{\mathcal{V}_G}(D(\mathcal{V}_G) + \epsilon\bar{d}) < F_{\mathcal{V}_G}(D(\mathcal{V}_G))$.

The following two facts are immediate.

C1.1: if we let $\bar{\delta}_1$ and $\bar{\delta}_i$ start from the same point on the reflectional axis of β, then the axis bisects the angle between these two vectors; and

C1.2: $|\bar{\delta}_1| = |\bar{\delta}_i|$.

Consequently, β is the reflectional symmetry of $\bar{D}(G)$ that induces q, where $\bar{D}(\mathcal{V}_G) = D(\mathcal{V}_G) + \epsilon\bar{d}$. Hence, $\bar{D}(G)$ displays P. Representative examples are illustrated in Figures 3(a) and 3(b).

Case 2:

First, we show that u_1 cannot be the fixed vertex (element) of P. On the contrary, suppose that u_1 is the fixed vertex of P. A set of directions d_j in $2n$-dimensional space are constructed as follows. For $0 \leq j \leq k-1$, $d_j = (\alpha^j\delta_1,\ (0,0)\ ...,\ (0,0))$. Note that $d_0 = d$, $\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d_0}\right)_{D(\mathcal{V}_G)} < 0$, and $p^i(u_1) = p^{i-1}(u_1) = u_1$ for $1 \leq i \leq k$. Iteratively applying Lemma 14, we have that for $1 \leq j \leq k-1$, $\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d_j}\right)_{D(\mathcal{V}_G)} < 0$.

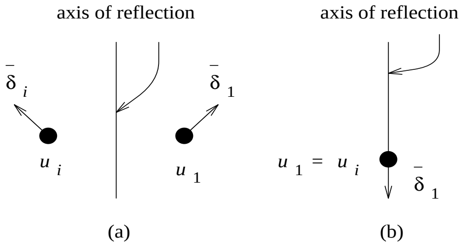

Figure 3: Illustration of Case 1

However, using the matrix representation (5) it is easy to verify that

$$d_0 = \begin{cases} -\frac{d_{(k-1)/2}+d_{(k+1)/2}}{|d_{(k-1)/2}+d_{(k+1)/2}|} & \text{if k is odd} \\ -d_{k/2} & \text{if k is even.} \end{cases}$$

From Lemma 13, we have $\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d_0}\right)_{D(\mathcal{V}_G)} > 0$. This contradicts our assumption. Hence u_1 cannot be the fixed vertex of P.

According to Proposition 1, the orbit W of P containing u_1 has size k and can be represented as $\{u_i : 1 \leq i \leq k\}$ where $u_i = p^{i-1}(u_1)$ for $1 \leq i \leq k$. We construct a set of directions $\{d_i^{'} : 1 \leq i \leq k\}$ as follows. For $1 \leq i \leq k$, $d_i^{'} = (\delta_{i,1}^{'}, \delta_{i,2}^{'}, ..., \delta_{i,n}^{'})$, where for $1 \leq i \leq k$, $1 \leq j \leq n$, $\delta_{i,j}^{'} = (0,0)$ if $i \neq j$, and $\delta_{i,i}^{'} = \alpha^{i-1}\delta_1$. Iteratively applying Lemma 14, we conclude that there is a $c < 0$ such that for $1 \leq i \leq k$, $\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d_i^{'}}\right)_{D(\mathcal{V}_G)} = c$.

Let $\bar{d} = \frac{\sum_{i=1}^{k} d_i^{'}}{|\sum_{i=1}^{k} d_i^{'}|}$. From Lemma 13, it follows that

$$\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial \bar{d}}\right)_{D(\mathcal{V}_G)} = \frac{kc}{|\sum_{i=1}^{k} d_i^{'}|} < 0.$$

Thus, there is an $\epsilon > 0$ such that $F_{\mathcal{V}_G}(D(\mathcal{V}_G) + \epsilon\bar{d}) < F_{\mathcal{V}_G}(D(\mathcal{V}_G))$. The following two facts are immediate:

C2.1: the angle from $\bar{\delta}_i$ to $\bar{\delta}_{i+1}$ is $\frac{2\pi}{k}$, and

C2.2: for $1 \leq i < j \leq k$, $|\bar{\delta}_i| = |\bar{\delta}_j|$.

Thus α is also the rotational symmetry of $\bar{D}(G)$ which induces p, where $\bar{D}(\mathcal{V}_G) = D(\mathcal{V}_G) + \epsilon\bar{d}$. Hence $\bar{D}(G)$ displays P. An example is illustrated in Figures 4(a).

Case 3:

From Lemma 2, this case consists of two subcases as follows.

Case 3 (a) q maps the orbit of $< p >$ containing u_1 to itself; and

Case 3 (b) q maps the orbit of $< p >$ containing u_1 to another orbit.

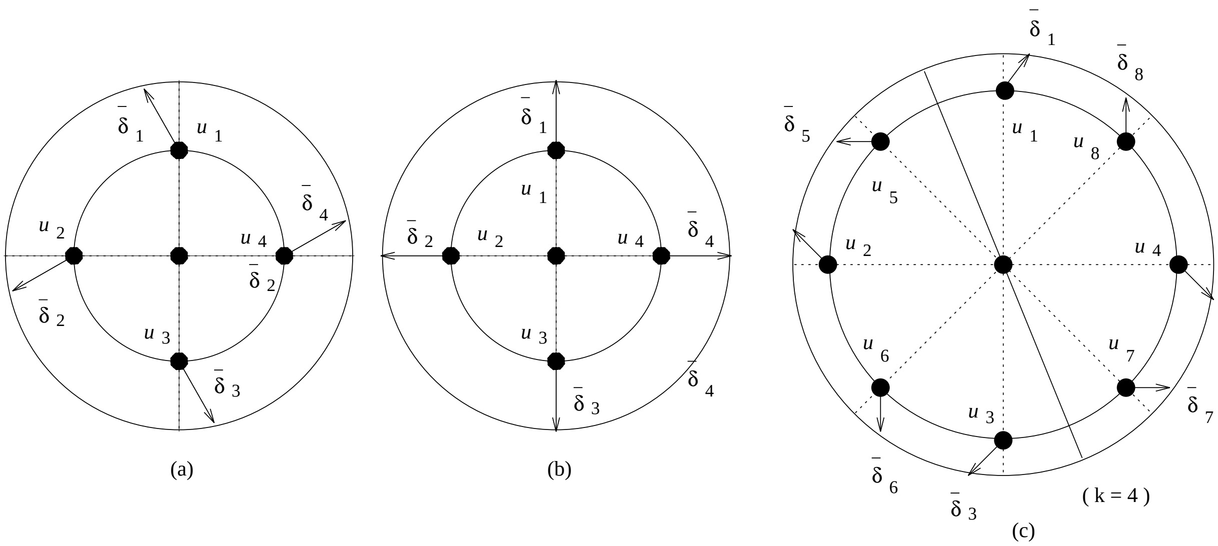

Figure 4: Illustration of Cases 2 and 3

As in Case 2, we can show that u_1 is not the fixed vertex of $< p >$. Let the orbit of $< p >$ containing u_1 be $W = \{u_1,\ u_2,\ ..., u_k\}$ where $u_j = p^{j-1}(u_1)$ $(1 \le j \le k)$.

Now we prove the Lemma for Case 3(a). Let $q(u_1) = u_{i_1}$ where $i_1 \in [1, k]$. By Proposition 1, for $1 \le j \le k$,

C3a.1: $q(u_j) = qp^{j-1}(u_1) = p^{k-j+1}(u_{i_1}) = p^{k-j+i_1}(u_1) = p^{i_1-j}(u_1) = u_{\{(i_1-j)\ mod\ k\}+1}$.

We construct a set $\{d_j\ :\ 1 \le j \le 2k\}$ of directions in $\Re^{2n}$ as follows. For $1 \le j \le 2k$, $d_j = (\delta_{j,1},\ \delta_{j,2},\ ...,\ \delta_{j,n})$ where for $1 \le z \le n$:

- $\delta_{j,z} = (0, 0)$ either when either $j \le k$ and $z \ne j$ or when $j \ge k+1$ and $z \ne j-k$, and
- $\delta_{j,j} = \alpha^{j-1}\delta_1$ if $j \le k$, and
- $\delta_{j,j-k} = \beta\alpha^{(i_1-(j-k))\ mod\ k}\delta_1$ if $j \ge k+1$ (noting C3a.1).

Iteratively applying Lemma 14, we conclude that there is $c < 0$ such that for $1 \le i \le 2k$, $\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d_i}\right)_{D(\mathcal{V}_G)} = c$. From Lemma 13, it follows that for $1 \le i \le k$, $\delta_{i,i} \ne -\delta_{i+k,i}$. Consequently, $\sum_{j=1}^{2k} d_j$ cannot be a zero vector. Let $\bar{d} = \frac{\sum_{j=1}^{2k} d_j}{|\sum_{j=1}^{2k} d_j|}$.

According to Lemma 13, $\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial \bar{d}}\right)_{D(\mathcal{V}_G)} = \frac{2kc}{|\sum_{j=1}^{2k} d_j|} < 0$. Thus, there is an $\epsilon > 0$ such that $F_{\mathcal{V}_G}(D(\mathcal{V}_G) + \epsilon\bar{d}) < F_{\mathcal{V}_G}(D(\mathcal{V}_G))$. The following two facts are immediate.

C3a.2: If the vectors $\bar{\delta}_i$ $(1 \le i \le k)$ start from the barycenter of $\{D(v)\ :\ v \in V\}$, the origin $(0, 0)$, then for $1 \le i \le k-1$, the angle from $\bar{\delta}_i$ to $\bar{\delta}_{i+1}$ is $\frac{2\pi}{k}$, and the reflectional axis of β bisects the angle between $\bar{\delta}_{i_1}$ and $\bar{\delta}_1$.

C3a.3: For $1 \le i < j \le k$, $|\bar{\delta}_i| = |\bar{\delta}_j|$.

These imply that α is also the rotational symmetry of $\bar{D}(G)$ inducing p where $\bar{D}(\mathcal{V}_G) = D(\mathcal{V}_G) + \epsilon\bar{d}$, and β is the reflectional symmetry of $\bar{D}(G)$ inducing q. Hence $\bar{D}$ displays P. This completes the proof for Case 3(a). A representative sample is illustrated in Figure 4(b).

Now we construct the proof for Case 3(b). Suppose that q map W to W'. According to Proposition 1, W' can be represented as $\{v_i : 1 \leq i \leq k\}$ such that $v_1 = q(u_1)$ and $v_j = p^{j-1}(v_1)$ for $1 \leq j \leq k$. Without loss of generality, we assume that $\mathcal{V}_G$ gives the vertex ordering such that $u_{k+j} = v_j$ for $1 \leq j \leq k$. Clearly, each vertex in W' is mapped from a vertex in W by q; that is, for $1 \leq j \leq k$, $u_{k+i_j} = q(u_j)$. Next, we compute the representation of i_j. By Proposition 1, for $1 \leq j \leq k$,

C3b.1: $q(u_j) = qp^{j-1}(u_1) = p^{k-j+1}(u_{k+1}) = u_{((k-j+1)\ mod\ k)+k+1}$.

Therefore $i_j = ((k-j+1)\ mod\ k) + 1$.

A set $\{d_j :\ 1 \leq j \leq 2k\}$ of directions in $2n$-dimensional space is constructed as follows. For $1 \leq j \leq 2k$, $d_j = (\delta_{j,1},\ \delta_{j,2},\ ...,\ \delta_{j,n})$ where for $1 \leq z \leq n$ (note $n \geq 2k$), $\delta_{j,z} = (0,0)$ if $z \neq j$, and

$$\delta_{j,j} = \begin{cases} \alpha^{j-1}\delta_1 & \text{if } j \leq k \\ \alpha^{j-k-1}\beta\delta_1 & \text{if } j \geq k+1 \text{ (noting C3b.1).} \end{cases}$$

Iteratively applying Lemma 14, we can conclude that there is a $c < 0$ such that for $1 \leq i \leq 2k$, $\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial d_i}\right)_{D(\mathcal{V}_G)} = c$.

Let $\bar{d} = \frac{\sum_{j=1}^{2k} d_j}{|\sum_{j=1}^{2k} d_j|}$. According to Lemma 13, $\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial \bar{d}}\right)_{D(\mathcal{V}_G)} = \frac{2kc}{|\sum_{j=1}^{2k} d_j|} < 0$. Thus, there is an $\epsilon > 0$ such that $F_{\mathcal{V}_G}(D(\mathcal{V}_G) + \epsilon\bar{d}) < F_{\mathcal{V}_G}(D(\mathcal{V}_G))$. The following two facts are immediate.

C3b.2: If the vectors $\bar{\delta}_i$ $(1 \leq i \leq 2k)$ start from the origin $(0,0)$, then for $1 \leq i \leq k-1$, the angle from $\bar{\delta}_i$ to $\bar{\delta}_{i+1}$ is $\frac{2\pi}{k}$, the angle from $\bar{\delta}_{k+i}$ to $\bar{\delta}_{k+i+1}$ is $\frac{2\pi}{k}$, and the reflectional axis of β bisects the angle between $\bar{\delta}_1$ and $\bar{\delta}_{k+1}$.

C3b.3: For $1 \leq i < j \leq 2k$, $|\bar{\delta}_i| = |\bar{\delta}_j|$.

Consequently, α is also the rotational symmetry of $\bar{D}(G)$ inducing p and β is the reflectional symmetry of $\bar{D}(G)$ inducing q, where $\bar{D}(\mathcal{V}_G) = D(\mathcal{V}_G) + \epsilon\bar{d}$. Hence $\bar{D}(G)$ displays P. A sample is in Figure 4(c). □

Below we show that Lemma 15 can be extended to cover the case where the total spring energy function is not differentiable at a given state.

Lemma 16 *Suppose that $D(G)$ is an unstable and finite energy state of a $\{\mathcal{F}, \mathcal{L}, G, D_0(\Omega)\}$ and displays P, where G has n vertices and P is a geometric automorphism group of G compatible with $D_0(\Omega)$. Further, suppose that the total spring energy function $F_{\mathcal{V}_G}$ is not differentiable at $D(\mathcal{V}_G)$. Then, there is a drawing $\bar{D}(G)$ such that $\bar{D}(\mathcal{V}_G) = D(\mathcal{V}_G) + \epsilon\bar{d}$ where $\bar{d}$ is a direction in $\Re^{2n}$ and $\epsilon > 0$; $F_{\mathcal{V}_G}(\bar{D}(\mathcal{V}_G)) < F_{\mathcal{V}_G}(D(\mathcal{V}_G))$; and $\bar{D}(G)$ displays P.*

Proof: As with the proof of Lemma 15, our proof divides into Case 1, Case 2, Case 3(a), Case 3(b). We assume that α and β are symmetries of $D(G)$, as described in the proof of Lemma 15, which induce p and q respectively. Without loss of generality, we assume that the barycenter of

$\{D(u) : u \in V\}$ is the origin $(0,0)$. Essentially the proof of this Lemma it to construct $\bar{d}$ with respect to these cases.

Note that $F_{\mathcal{V}_G}$ is not differentiable at $D(\mathcal{V}_G)$. From Lemma 6, there is a set U of m ($m \geq 2$) vertices, $U = \{u_{z_1}, u_{z_2}, ..., u_{z_m}\}$, such that:

> for every $u_{z_j} \in U$, there is at least a vertex $u_{z_i} \in U$ where $f_{u_{z_j} u_{z_i}} \not\equiv 0$, $D(u_{z_j}) = D(u_{z_i})$, and $l_{u_{z_j} u_{z_i}} > 0$.

Since $D_0(\Omega)$ is a proper drawing, at least one vertex in U is not in Ω; that is, $U - \Omega \neq \emptyset$. Let $\bar{U}$ denote $U - \Omega$.

Now we show that in both Case 2 and Case 3, there is a vertex v in $\bar{U}$ such that v is not the fixed vertex of $< p >$. This is trivial when $|\bar{U}| \geq 2$, because $< p >$ has at most one fixed vertex. So, we need only to prove that if $|\bar{U}| = 1$, then the element of $\bar{U}$ is not the fixed vertex of $< p >$.

On the contrary, suppose that there is only one vertex w in $\bar{U}$ and w is the fixed vertex of $< p >$. Thus, there is a vertex u in $\Omega \cap U$ such that $D(u) = D(w)$. Note that $D(G)$ displays P. Therefore, $D(w)$ must be placed at $(0,0)$. Since P is compatible with $D_0(\Omega)$, the k vertices of the orbit of $< p >$ containing u are in Ω and are all located at $D(w)$. This contradicts the fact that $D_0(\Omega)$ is proper. Thus, we can conclude that there is a vertex v in $\bar{U}$ which is not the fixed vertex of $< p >$ in Case 2 and Case 3.

Without loss of generality, we may assume that vertex u_1 in the sequence $\mathcal{V}_G$ is such a vertex that $u_1 \in \bar{U}$ and is not the fixed vertex of $< p >$ with respect to Case 2 and Case 3. We choose a 2-dimensional direction δ_1 respectively in each case such that

- in Case 1, $\beta\delta_1 \neq \delta_1$ and $\beta\delta_1 \neq -\delta_1$;
- in Case 3(a), $\beta\delta_1 \neq -\alpha^{i_1-1}\delta_1$ where $p(u_1) = u_{i_1}$;
- in Case 2 and Case 3(b), δ_1 is an arbitrary direction in $\Re^2$.

Let $\bar{d} = (\bar{\delta}_1, \bar{\delta}_2, \ldots, \bar{\delta}_n)$. With respect to u_1, δ_1, Case 1, Case 2, Case 3(a), and Case 3(b), one may immediately verify that it is feasible to construct $\bar{d}$ respectively in the same way as described in the proof of Lemma 15.

We may verify that C1.1, C1.2, C2.1, C3a.1, C3a.2, C3a.3, C3b.1, C3b.2, C3b.3 are respectively satisfied. Hence, the application of Theorem 3 yields that respectively in each case either

$$\left(\frac{\partial F_{\mathcal{V}_G}(X)}{\partial \bar{d}}\right)_{D(\mathcal{V}_G)} < 0,$$

or

$$\left(\frac{\partial F_{\mathcal{V}_G}(D(X))}{\partial (-\bar{d})}\right)_{D(\mathcal{V}_G)} < 0.$$

Without loss of generality, we may assume that for each case,

$$\left(\frac{\partial F_{\mathcal{V}_G}(D(\mathcal{V}_G))}{\partial \bar{d}}\right)_{D(\mathcal{V}_G)} < 0.$$

Thus, there is an $\epsilon > 0$ such that $F_{\mathcal{V}_G}(D(\mathcal{V}_G) + \epsilon\bar{d}) < F_{\mathcal{V}_G}(D(\mathcal{V}_G))$, and the drawing $\bar{D}(G)$, with $\bar{D}(\mathcal{V}_G) = D(\mathcal{V}_G) + \epsilon\bar{d}$, displays P with respect to all these cases. $\square$

4.4 Convergence to a Stable State

In this subsection, we detail Step 3 in the proof of Theorem 1.

Given a set $B = \{X_i : 1 \leq i < \infty\}$ of infinite points in the n-dimensional Euclidean space $\Re^n$, a point X is a *convergent point* of B if there is a subsequence $\{X_{i_j} : 1 \leq j \leq \infty\}$ of B such that

$$\lim_{j \to \infty} X_{i_j} = X.$$

A set C of points is *closed* if for every infinite sequence B in S, all convergent points of B are in S. For a continuous function $F : \Re^n \to +\Re$, the *greatest lower bound* of C with respect to F is a real number c, such that

- for every element $X \in C$, $F(X) \geq c$, and
- for every $\epsilon > 0$, there is an element X in C such that $F(X) < c + \epsilon$.

Consequently, the following Lemma is well-known true [21] in the contents of calculus for a closed set in $\Re^n$.

Lemma 17 *Suppose that C is a closed set of points in $\Re^n$, F is a continuous function where $F : \Re^n \to +\Re$, c is the greatest lower bound of C with respect to F. Then, there is an element $X \in C$ such that $F(X) = c$.*

The following Theorem says that a convergent point of a sequence of drawings preserves symmetries.

Theorem 4 *Suppose that $\{D_i(G) : 1 \leq i < \infty\}$ is a sequence of drawings of a graph G which converge to a drawing $D(G)$ of G as $i \to \infty$, that is,*

$$D(\mathcal{V}_G) = \lim_{i \to \infty} D_i(\mathcal{V}_G).$$

Further, suppose that p is a geometric automorphism of G. If each $D_i(G)$ displays p, then $D(G)$ also displays p.

Proof: We prove this Theorem with respect to the two types of geometric automorphism,

case 1: p is a reflectional automorphism, or

case 2: p is a k-rotational automorphism.

In case 1, if $D(G)$ locates all vertices at the same point in the plane then the Theorem is trivially true; and thus, we need only to prove case 1 when $D(G)$ *does not locate all vertices at the same point* . There are two subcases:

Case 1(a) For some vertex u_0, $D(u_0) \neq D(p(u_0))$.

In this case let ζ be the perpendicular bisector of the line segment from $D(u_0)$ to $D(p(u_0))$. Since $D_i(G)$ is convergent to $D(G)$, there is an integer K_0 such that for $i \geq K_0$, $D_i(u_0) \neq D_i(p(u_0))$, so that ζ_i may be chosen for $D_i(G)$ in the same way as ζ for $D(G)$.

Case 1(b) For all vertices u, $D(u) = D(p(u))$.

Because we assume that $D(G)$ does not locate all vertices at the same point, there must be at least one vertex u_1 such that $D(u_1)$ is not the barycentre of $\{D(v) : v \in V\}$. Let ζ be a line through $D(u_1)$ and the barycentre. Since $D_i(G)$ converges to $D(G)$, there is an integer K_1 such that for $i \geq K_1$, $D_i(G)$ is either in case 1(a) or in case 1(b). We choose ζ_i ($i \geq K_1$) for $D_i(G)$ in the same way as ζ for $D(G)$ with respect to either case 1 (a) or case 1(b).

Choose $K = \max(K_0, K_1)$. From the fact that each $D_i(G)$ displays p, it can be verified that for $i \geq K$, the reflection β_i with axis ζ_i must satisfy RE2 and RE3 with respect to $D_i(G)$. Using the the fact that $D_i(G)$ is convergent to $D(G)$, one can immediately verify that the reflection β with axis ζ also satisfies RE2 and Re3 with respect to $D(G)$. Therefore, β induces p and $D(G)$ displays p.

In case 2, we may choose the barycentre of $\{D(v) : v \in V\}$ as X; and for each i, choose the barycentre of $\{D_i(v) : v \in V\}$ as X_i. Note that each $D_i(G)$ displays p. Consequently, for each $D_i(G)$ there must be a rotation α_i by $\frac{2\pi}{k}$ with the center X_i such that RO2 and RO3 are satisfied. Using the fact that $D_i(G)$ converges to $D(G)$, one can immediately verify that the rotation α by $\frac{2\pi}{k}$ at X satisfies RO2 and RO3 with respect to $D(G)$. Hence α induces p and $D(G)$ displays p. $\square$

Now we are ready to prove Theorem 1.

Proof of Theorem 1: Suppose that $V - \Omega \neq \emptyset$; otherwise the Theorem is trivially true as P is compatible with $D_0(\Omega)$. According to Lemma 11, there is at least one proper drawing $D^*(G)$ which displays P and is a state of the general spring system. Since $D^*(G)$ is proper, $D^*(G)$ is a finite energy state.

Let $\mathcal{V}_G$ give the vertex ordering such that $u_1 \in \Omega$ when $\Omega \neq \emptyset$; otherwise, u_1 is just the first vertex in $\mathcal{V}_G$. Let K denote $F_{\mathcal{V}_G}(D^*(\mathcal{V}_G))$. Since $D^*(G)$ is a finite state, $K \geq 0$ but $K \neq \infty$. Note that from the definition of a spring energy function, it follows that for every spring energy function f_{uv} such that $f_{uv} \not\equiv 0$,

$$\lim_{x \to +\infty} f_{uv}(x) = \infty.$$

We can choose a positive real number η such that

- for every nonzero spring energy function f_{uv}, if $x > \eta$ then $f_{uv}(x) > K$; and
- for every pair of vertices u, v in Ω, we have $|D_0(u) - D_0(v)| < \eta$.

Let S denote the set of vectors (points) in $2n$-dimensional space such that $\forall((x_1, y_1),\ (x_2, y_2),\ \ldots,\ (x_n, y_n)) \in S$, $(x_1, y_1) = D^*(u_1)$, and

- for $1 \leq i \leq n$, $|(x_i, y_i) - D^*(u_1)| \leq n(\eta + \sum_{\{u,v\} \subset V, u \neq v, l_{uv} \neq \infty} l_{uv})$, and
- for every pair of $\{i, j\}$ where $1 \leq i, j \leq n$ such that $l_{u_i u_j} = \infty$, $|(x_i, y_i) - (x_j, y_j)| \geq \frac{1}{\eta}$,

where all (x_i, y_i) are 2-dimensional vectors (points). Let $S_{G,P,\Omega}$ denote the following set:

$$\{\tilde{D}(\mathcal{V}_G) :\ \tilde{D}(G) \text{ is a state of the spring system, } \tilde{D}(G) \text{ displays } P, \text{ and } \tilde{D}(\mathcal{V}_G) \in S\}.$$

Immediately, we can verify that for a drawing $\tilde{D}(G)$, if $\tilde{D}(u_1) = D^*(u_1)$ and $\tilde{D}(\mathcal{V}_G) \notin S$ then $F_{\mathcal{V}_G}(D(\mathcal{V}_G)) > K$. Clearly, $S_{G,P,\Omega}$ is not empty because $D^*(\mathcal{V}_G)$ is in $S_{G,P,\Omega}$.

Further, it can be immediately verified that S is a bounded set, and thus a closed set. This together with Theorem 4 imply that $S_{G,P,\Omega}$ is closed.

Suppose that c is the greatest lower bound of $S_{G,P,\Omega}$ with respect to $F_{\mathcal{V}_G}$. It implies that $0 \leq c \leq K$. From the fact $F_{\mathcal{V}_G}$ is continuous on S and Lemma 17, it follows that there is a $D(\mathcal{V}_G) \in S_{G,P,\Omega}$ such that $F_{\mathcal{V}_G}(D(\mathcal{V}_G)) = c$.

We can also conclude that $D(G)$ is stable. If the contrary holds, then by Lemmas 15 and 16, there is a state $D'(G)$ of the spring system which displays P such that $F_{\mathcal{V}_G}(D'(\mathcal{V}_G)) < F_{\mathcal{V}_G}(D(\mathcal{V}_G)) \leq C$. Note that a drawing $\tilde{D}(G)$ such that $\tilde{D}(\mathcal{V}_G) \notin S$ and $\tilde{D}(u_1) = D^*(u_1)$ satisfies the condition $F_{\mathcal{V}_G}(\tilde{D}(\mathcal{V}_G)) > K \geq c$. Consequently, $D'(\mathcal{V}_G) \in S_{G,P,\Omega}$ if $\Omega \neq \emptyset$. When $\Omega = \emptyset$, using a 2-dimensional translation α such that $\alpha D'(u_1) = D^*(u_1)$, we can conclude that $F_{\mathcal{V}_G}(\alpha D'(\mathcal{V}_G)) = F_{\mathcal{V}_G}(D'(\mathcal{V}_G)) < F_{\mathcal{V}_G}(D(\mathcal{V}_G))$. Thus, $\alpha D'(\mathcal{V}_G) \in S_{G,P,\Omega}$. These contradict the fact that $F_{\mathcal{V}_G}(D(\mathcal{V}_G))$ is the greatest lower bound of $S_{G,P,\Omega}$ with respect to $F_{\mathcal{V}_G}$. Thus $D(G)$ is stable. □

5 Conclusions

In this paper, firstly we characterized graph symmetric properties, displayed by a 2-dimensional drawing. We showed that the representation of graph symmetries as a geometric automorphism group is complete. Secondly, we proved that for a geometric automorphism group P, many existing spring algorithms are always possible to simultaneously display the geometric automorphisms in P as the symmetries of a drawing produced by A.

As we mentioned, there are some spring systems which are not covered by this paper. However these systems are not specifically designed for producing symmetric drawings.

Note that a spring algorithm usually randomly chooses a drawing in its drawing space. In this paper we showed that graph symmetries are always possible to be displayed by a spring algorithm. It shall be also interesting to estimate the possibility to produce a symmetric drawing. To do this, the following two issues should be investigated first:

1. What is the cardinality of the set of stable states for a given spring system and graph?

2. How do you measure the difference between two stable states?

These will be our future study.

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